Special Session on Semigroups, Algorithms and Universal Algebra

AMS Meeting, Louisville, March 20–21, 1998

Problems Submitted

Algorithmic Problems

The problems in "input/output" format are decision problems. What is wanted is a proof of existence/non-existence of an algorithm (Turing machine) that accepts the indicated input data and always outputs the required output data.

(1) (Mark Sapir) Input: (1) A finite set of equations in a finite type, including Jónsson equations, so that the variety $\mathcal{V}$ axiomatized by this set is congruence-distributive. (2) A finite algebra $\mathbf{A}$ of the same type. Output: Decide if $\mathbf{A}$ is embeddable into a finite simple algebra in $\mathcal{V}$. [This is undecidable for non-congruence-distributive varieties.]

(2) (George McNulty) Say that a finite algebra $\mathbf{A}$ is *expandably residually finite* if $\mathbf{A}$ has finite type and every finite expansion $\mathbf{A}'$ (obtained by adding finitely many operations to $\mathbf{A}$) has $\mathsf{V}(\mathbf{A}')$ residually finite. Is the class of expandably residually finite algebras recursive? Find an intrinsic characterization of this class of algebras.

(3) (Steve Seif) Input: A finite semigroup $\mathbf{S}$. Output: Decide if the term-equivalence problem for $\mathbf{S}$ is in P. (If $P = NP$, this is easy.)

(4) ("the luncheon group") Input: (1) Finite algebra $\mathbf{A}$ in a finite language. (2) $n \in \omega$. Output: Decide if $\mathsf{V}(\mathbf{A})$ is residually $\leq n$.

The next nine problems were submitted by Ralph McKenzie.

(5) Is there a recursive class $\mathcal{A}$ of finite algebras of finite type such that $\mathrm{typ}(\mathsf{V}(\mathbf{F})) = \{4\}$ implies $\mathbf{F} \in \mathcal{A}$ and $\mathrm{typ}(\mathsf{V}(\mathbf{F})) = \{5\}$ implies $\mathbf{F} \notin \mathcal{A}$? I.e., is the property $\mathrm{typ}(\mathsf{V}(\mathbf{F})) = \{4\}$ recursively separable from the property $\mathrm{typ}(\mathsf{V}(\mathbf{F})) = \{5\}$? Same question for type-sets $\{1,3\}$ and $\{1,5\}$, and for type-sets $\{1,3\}$ and $\{1,2,3\}$. [The "type-sets" involved here are those defined in D. Hobby, R. McKenzie, *The Structure of Finite Algebras*. With the answers to these questions, we could complete a list of all the recursive properties of finite algebras $\mathbf{F}$ which are determined by the type-set of $\mathsf{V}(\mathbf{F})$.]

(6) Is the class of all finite, equationally finitely-based algebras, of finite type, a recursively enumerable class?

(7) Input: (1) A finite algebra $\mathbf{F}$ of finite type. (2) A finite list of equations holding in $\mathbf{F}$ showing that $\mathsf{V}(\mathbf{F})$ is congruence-modular. output: Decide if $\mathsf{V}(\mathbf{F})$ is finitely axiomatizable.

(8) (A. Tarski, P. Perkins, M. Sapir) Input: (1) A finite semigroup $\mathbf{S}$. Output: Decide if $\mathsf{V}(\mathbf{S})$ is finitely axiomatizable.

(9) Write $\kappa(\mathbf{F})$ for the least cardinal exceeding the size of every subdirectly irreducible algebra in $\mathsf{V}(\mathbf{F})$, if there is such a cardinal, and put $\kappa(\mathbf{F}) = \infty$ if there is no such cardinal. Here are three problems. Input: A finite algebra $\mathbf{F}$ of finite type with $5 \notin \mathrm{typ}(\mathsf{V}(\mathbf{F}))$. (i) Output: decide if $\mathsf{V}(\mathbf{F})$ is finitely axiomatizable. (ii) Output: decide if $\kappa(\mathbf{F}) < \omega$. (iii) Output: Decide if $\kappa(\mathbf{F}) < \infty$.

(10) In connection with the preceeding problems, does there exist a finite algebra (possibly with infinitely many operations) with $5 \notin \mathrm{typ}(\mathsf{V}(\mathbf{F}))$ and $\omega \leq \kappa(\mathbf{F}) < \infty$?

(11) Input: A finite algebra $\mathbf{F}$ in a finite language. Output: Decide if $\mathbf{F}$ has finitely based quasi-equations.

(12) Input: A finite algebra $\mathbf{F}$ in a finite language. Output: Decide if there exists some finitely axiomatizable, locally finite quasi-variety $\mathcal{K}$ containing $\mathsf{SP}(\mathbf{F})$.

(13) Input: A finite algebra $\mathbf{F}$ in a finite language. Output: Decide if $\mathbf{F}$ has a term operation $t(x_1, \ldots, x_n)$ satisfying the near-unanimity equations $t(x, \ldots, x, y, x, \ldots, x) \approx x$, for some $n \geq 3$. [If this is undecidable, then it is undecidable to determine if $\mathsf{SP}(\mathbf{F})$ admits a natural duality in the sense of Davey and Priestley.]

Other Problems

(1) (Victor Gorbunov, submitted by Mark Sapir) Given a finite algebra, is it true that it either is finitely based or has no independent basis of equations? (Same problem for quasi-equations.)

(2) (B.H. Neumann, submmited by Constantine Tsinakis) Can every orderable group be embedded into a divisible orderable group?

(3) (R.W. Quackenbush, submitted by Ralph McKenzie) Prove or disprove: If $\mathbf{A}$ is a finite algebra in a finite language and $\mathsf{V}(\mathbf{A})$ is residually finite, then $\mathsf{V}(\mathbf{A})$ has a finite residual bound.

(4) (J. Ježek, P. Markovic, M. Maroti, R. McKenzie) A tournament is a groupoid $\langle A, \cdot \rangle$ satisfying $xy = yx \in \{x, y\}$ for all $x, y \in A$. Let $\mathcal{T}$ be the variety generated by all tournaments, $\mathcal{T}_n$ be the variety generated by all n-element tournaments. Conjecture: Every subdiectly irreducible algebra in $\mathcal{T}$ is a tournament. Problem: Is $\mathcal{T}$ inherently nonfinitely based? Problem: Is $\mathcal{T}_n$ residually finite?

(5) (Ralph Freese) Conjecture: A finitely presented lattice is infinite iff it has an element which is join-irreducible but not completely join-irreducible, or meet-irreducible but not completely meet-irreducible.

(6) (Garrett Birkhoff, 1942; submitted by Jonathan Farley) Conjecture: Let $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ be finite posets. If $\mathbf{Q}^{\mathbf{P}} \cong \mathbf{R}^{\mathbf{P}}$ then $\mathbf{Q} \cong \mathbf{R}$. (Here $\mathbf{Q}^{\mathbf{P}}$ is the poset of all monotone mappings from $\mathbf{P}$ into $\mathbf{Q}$, ordered as a subset of the Cartesian power $\mathbf{Q}^{P}$.)

(7) (S. Bulman-Fleming, submitted by Matthew Gould and Mick Adams) For a finite poset **P**, the *Aiženstāt-Howie property* is the property that every non-bijective order-preserving selfmap is a composition of idempotent order-preserving selfmaps. Results of Aiženstät and Howie (individually) imply that the following posets have the A-H property: (a) finite chains; (b) finite antichains; (c) finite antichains with 0 or 1 adjoined; (d) finite antichains with 0 and 1 adjoined. <u>Conjecture:</u> These are the only finite posets that have the A-H propert. [Reference: Adams, Bulman-Fleming, Gould, and Wildsmith, Proc. Royal Soc. Edinburgh, circa 1994.]

(8) (Matthew Valeriote) Characterize those finite posets **P** such that the variety generated by the order primal algebra $\mathbf{A}(\mathbf{P})$ (A) satisfies some nontrivial congruence equation; or (B) omits types 1 and 5.

(9) (Dejan Delic) <u>Problem:</u>

1. 1. Does DOPC imply DPC for M-algebras?
2. 2. Give a characterization of those finite M-algebras $\mathbf{A}$ such that $\mathsf{V}(\mathbf{A})$ has DOPC (DPC).
3. 3. Does $\mathsf{SP}(\mathbf{A})$ finitely axiomatizable imply $\mathsf{V}_{SI}(\mathbf{A}^\wedge)$ finitely axiomatizable?

(10) (George McNulty) Is there a finite algebra $\mathbf{A}$ which is not dualizable but $\mathbf{A} \in \mathsf{SP}(\mathbf{B})$ for some finite dualizable algebra $\mathbf{B}$?

(11) (George McNulty) Describe the equational theory of

$$\langle \omega, x+y, x \cdot y, x!, C(x,y), 0, 1 \rangle$$

where $C(x,y)$ is the binomial coefficient.

(12) (Steve Seif) Let X be a set and let $\mathbf{S}$ be a semigroup of transformations of X. Then $\mathbf{S}$ is said to act primitively on X if the unary algebra $(X; S)$ is simple (in the universal algebra sense; that is, if the only $\mathbf{S}$-compatible equivalence relations on X are the diagonal and universal relations). Assuming that X has more than 2 elements, does there exist such a primitive action if $\mathbf{S}$ satisfies the following "nilpotence" condition: for all $s \in S$, there exists a positive integer n such that s^n is a constant function? If S is finite, the answer is "no."

(13) (Alden Pixley, submitted by Ralph McKenzie) Assume that V is a residually finite variety that is not congruence-distributive. Must there exist a finite algebra in V with non-distributive congruence lattice?